

# Hidden Symmetries and Geodesics of Kerr spacetime in Kaluza-Klein Theory

Alikram N. Aliev

*Yeni Yüzyıl University, Faculty of Engineering and Architecture,  
Cevizlibağ-Topkapı, 34010 Istanbul, Turkey*

Göksel Daylan Esmer

*Istanbul University, Department of Physics, Vezneciler, 34134 Istanbul, Turkey*

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## Abstract

The Kerr spacetime in Kaluza-Klein theory describes a rotating black hole in four dimensions from the Kaluza-Klein point of view and involves the signature of an extra dimension that shows up through the appearance of the electric and dilaton charges. In this paper, we study the separability properties of the Hamilton-Jacobi equation for geodesics and the associated hidden symmetries in the spacetime of the Kerr-Kaluza-Klein black hole. We show that the complete separation of variables occurs only for *massless* geodesics, implying the existence of hidden symmetries generated by a second rank *conformal* Killing tensor. Employing a simple procedure built up on an “effective” metric, which is conformally related to the original spacetime metric and admits a complete separability structure, we construct the explicit expression for the conformal Killing tensor. Next, we study the properties of the geodesic motion in the equatorial plane, focusing on the cases of static and rotating Kaluza-Klein black holes separately. In both cases, we obtain the defining equations for the boundaries of the regions of existence, boundedness and stability of the circular orbits as well as the analytical formulas for the orbital frequency, the radial and vertical epicyclic frequencies of the geodesic motion. Performing a detailed numerical analysis of these equations and frequencies, we show that the physical effect of the extra dimension amounts to the significant enlarging of the regions of existence, boundedness and stability towards the event horizon, regardless of the classes of orbits.

## I. INTRODUCTION

As is known, one of the most attractive features of the ordinary Kerr spacetime, which describes a family of rotating black holes in general relativity (GR), is its separability structure. The Hamilton-Jacobi equation for geodesics admits a complete separation of variables [1], despite the fact that the spacetime possesses only two global isometries generated by two commuting Killing vector fields. Clearly, the separability structure implies an extra integral of motion that in turn signals the existence of hidden symmetries in the spacetime. The authors of [2] showed that this is indeed the case. The Kerr spacetime possesses hidden symmetries generated by a second rank symmetric Killing tensor, rendering the Hamilton-Jacobi equation for geodesics completely integrable. Moreover, the Kerr spacetime provides full quantum separability in both the Klein-Gordon equation [3] and the Dirac equation [4, 5]. While the existence of the Killing tensor plays a crucial role for the separability of the Klein-Gordon equation, the situation with the Dirac equation is more subtle. In addition to the Killing tensor, the Kerr spacetime also admits a second rank antisymmetric Killing-Yano tensor which can be thought of as a “square root” of the Killing tensor [6]. It is the Killing-Yano tensor that lies behind the separability of the Dirac equation in the Kerr background [7]. Thus, in a sense, the usual “square root” relationship between the Dirac and Klein-Gordon equations turns out to be echoed in the structure of hidden symmetries of the Kerr spacetime.

In recent years, there has been an active interest in hidden symmetries of higher-dimensional black hole spacetimes. In [8, 9], it was shown that the spacetime of higher-dimensional rotating black holes given by the Myers-Perry metric [10], which generalizes the Kerr metric to all higher dimensions, admits both the Killing and Killing-Yano tensors. In other words, the hidden symmetries of the Kerr spacetime survive for the Myers-Perry spacetime in higher dimensions as well. To gain some insight into the origin of hidden symmetries generated by the Killing-Yano tensor, the authors of [11] managed to relate them to a new kind of *supersymmetry*, appearing in the worldline supersymmetric mechanics of spinning point particles in the Kerr background. In a recent work [12], it was shown that similar analysis based on the viewpoint of worldline supersymmetric mechanics also remains true for the higher-dimensional spacetime of the Myers-Perry black holes. The hidden symmetries of general rotating charged black holes in five-dimensional minimal gauged

supergravity [13] as well as various black hole solutions of supergravity and string theories have been studied in a number of works (see, for instance, [14–19] and references therein).

Intriguing generalizations of the Kerr spacetime have also been studied in Kaluza-Klein theory. In a relatively simple setting, the Kerr solution in Kaluza-Klein theory describes a rotating black hole in four dimensions from the Kaluza-Klein point of view and involves the signature of an extra dimension. This shows up through the appearance of the electric and dilaton charges, though the dilaton charge is not an independent parameter. That is, the solution satisfies the coupled Einstein-Maxwell-dilaton field equations, which are obtained from the Kaluza-Klein reduction of Einstein gravity in five dimensions. The procedure of obtaining such a solution is well known [20–22] and amounts to boosting a four-dimensional “seed” solution under consideration in the fifth dimension with a subsequent Kaluza-Klein reduction to four dimensions. The most general solution for rotating black holes in Kaluza-Klein theory was obtained in [23, 24] by employing a solution generation technique based on the use of hidden symmetries of the Einstein field equations.

Recently, intriguing developments have also been towards exploring the physical effects of black holes in four and higher dimensions. Observations of rapidly rotating black holes (with the angular momenta approaching the Kerr bound in GR) in some X-ray binaries [25, 26] have sparked the old theoretical question of *bona fide spacetime geometry* around the black holes. In light of this, many investigators have studied gravitational effects of black holes both in GR and beyond it, focusing in some cases on the imprints of the extra dimension in our physical world (for instance, see Refs.[27–34] and references therein).

The purpose of the present paper is two-fold: Firstly, we examine the separability structure and the hidden symmetries of the rotating black hole in the Kaluza-Klein framework (the Kerr-Kaluza-Klein black hole), where it carries the imprint of the extra fifth dimension through the electric and dilaton charges. We consider the Hamilton-Jacobi equation for a massive (uncharged) particle in the background of this black hole and show that the complete separation of variables occurs only for the vanishing mass of the particle, in contrast to the case of the original Kerr black hole in GR. This implies that the black hole spacetime under consideration possesses hidden symmetries generated by a second rank conformal Killing tensor. Next, we construct the explicit form for the conformal Killing tensor by employing a nice procedure built up on an effective metric, which is conformally related to the original spacetime metric and admits the separability structure due to the Killing

tensor. Such a procedure of constructing the conformal Killing tensor was earlier used in [35] as well. Secondly, we explore the geodesic motion of the uncharged massive particle in the equatorial plane of the Kerr-Kaluza-Klein black hole. Using the Hamilton-Jacobi and geodesic equations, we study the effects of the extra fifth dimension on the properties of the circular motion around this black hole. We show that the extra dimension has its greatest effect in enlarging the regions of existence, boundedness and stability of the circular motion towards the event horizon, regardless of the classes of orbits.

The outline of the paper is as follows: In Sec.II we describe a theoretical framework for the Kerr-Kaluza-Klein black hole. This includes a brief recalling the construction of the pertaining spacetime metric, the description of its physical properties as well as the properties of the Hamilton-Jacobi equation in this spacetime. In Sec.III we introduce a procedure that builds up on the use of an effective metric, conformally related to the original spacetime metric and admitting the Killing tensor. Here we show that such a procedure enables one to construct the conformal Killing tensor for the Kerr-Kaluza-Klein spacetime under consideration. In Sec.IV we study the properties of the circular and quasicircular (epicyclic) motions in the equatorial plane of both static and rotating Kaluza-Klein black holes. In both cases, we present defining equations for the boundaries of the existence regions as well as for the boundaries of the regions of boundedness and stability of the circular motion. Here we also present the results of a detailed numerical analysis of these equations. Next, using the general theory of the epicyclic motion, earlier developed in [36, 37], we give the analytical expressions for the orbital, radial and vertical epicyclic frequencies and perform the numerical analysis of these expressions. In Sec.V we conclude with the discussion of our results.

## II. THE KERR-KALUZA-KLEIN BLACK HOLE

We begin by recalling briefly the construction of the exact solution that represents a rotating black hole in Kaluza-Klein theory, namely the Kerr-Kaluza-Klein black hole with the Maxwell and dilaton fields. The details of the construction can be found in the original paper [21] as well as in a recent paper [22], including a NUT parameter as well. At the first step, the procedure of obtaining this solution amounts to adding an extra spacelike flat dimension to the usual Kerr solution of four-dimensional GR. Thus, in the Boyer-Lindquist

coordinates we have the five-dimensional metric given by

$$ds_5^2 = -\frac{\Delta}{\Sigma} \left( dt - a \sin^2 \theta d\phi \right)^2 + \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2 \theta}{\Sigma} \left[ a dt - (r^2 + a^2) d\phi \right]^2 + dy^2, \quad (1)$$

where

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad (2)$$

the parameters  $M$  and  $a$  determine the mass and the angular momentum of the solution.

Next, one needs to boost this metric in the fifth dimension by

$$\begin{aligned} t &\rightarrow t \cosh \alpha + y \sinh \alpha \\ y &\rightarrow y \cosh \alpha + t \sinh \alpha, \end{aligned} \quad (3)$$

and with the velocity of the boost  $v = \tanh \alpha$ . Clearly, the boosted metric will satisfy the vacuum equations of five-dimensional GR. Putting this metric into the standard Kaluza-Klein form

$$ds_5^2 = e^{-2\Phi/\sqrt{3}} ds_4^2 + e^{4\Phi/\sqrt{3}} (dy + 2A)^2, \quad (4)$$

we compactify the extra fifth dimension, identifying the four-dimensional metric

$$\begin{aligned} ds_4^2 = & -\frac{1}{B} \frac{\Delta}{\Sigma} \left( dt - a \cosh \alpha \sin^2 \theta d\phi \right)^2 + B \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right) - \frac{\Delta \sin^2 \theta}{B} \sinh^2 \alpha d\phi^2 \\ & + \frac{\sin^2 \theta}{B \Sigma} \left[ a dt - (r^2 + a^2) \cosh \alpha d\phi \right]^2, \end{aligned} \quad (5)$$

and the associated potential one-form  $A$  and the dilaton field  $\Phi$ , which are given by

$$A = \frac{Z \sinh \alpha}{2B^2} \left( \cosh \alpha dt - a \sin^2 \theta d\phi \right), \quad \Phi = \frac{\sqrt{3}}{2} \ln B. \quad (6)$$

Here we have used the notation

$$B = \left( 1 + \frac{2Mr \sinh^2 \alpha}{\Sigma} \right)^{1/2}. \quad (7)$$

We see that for the vanishing boost velocity,  $\alpha \rightarrow 0$ , the Maxwell and dilaton fields vanish and the metric in (5) reduces to the original Kerr solution. It is straightforward to check that solution (5), accompanied with the Maxwell and dilaton fields given in (6), satisfies the equation of motion derived from the four-dimensional action of Kaluza-Klein theory

$$S = \int d^4x \sqrt{-g} \left[ R - 2(\partial\Phi)^2 - e^{2\sqrt{3}\Phi} F^2 \right], \quad (8)$$

where  $F = dA$ . We recall that this action is obtained from the five-dimensional Einstein action for the metric in the form given by (4). (See Ref.[22] for details).

### A. Physical Properties

It is easy to see that the spacetime in (5) admits two commuting Killing vectors  $\xi_{(t)} = \partial/\partial t$  and  $\xi_{(\phi)} = \partial/\partial \phi$ , which reflect its time-translational and rotational invariance. Calculating the various scalar products of these vectors, we arrive at the metric components in the form

$$\begin{aligned}\xi_{(t)} \cdot \xi_{(t)} &= g_{00} = -\frac{1}{B} \left(1 - \frac{2Mr}{\Sigma}\right), \\ \xi_{(t)} \cdot \xi_{(\phi)} &= g_{03} = -\frac{2Mra \sin^2 \theta}{B\Sigma} \cosh \alpha, \\ \xi_{(\phi)} \cdot \xi_{(\phi)} &= g_{33} = \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{B^2 \Sigma}\right) B \sin^2 \theta.\end{aligned}\tag{9}$$

On the other hand, as follows from metric (5), the boosting and dimensional reduction procedures do not change the location of the event horizon. It is still determined by the largest root of the equation  $\Delta = 0$ , which is given by

$$r_+ = M + \sqrt{M^2 - a^2},\tag{10}$$

implying that the horizon exists provided that  $a \leq M$ . As for the physical parameters of the metric, the total mass, angular momentum and the total electric charge, they can be determined by evaluating the corresponding Komar integrals and the flux integral over a 2-sphere at spatial infinity, respectively. This has been done in works [20–22]. Writing these parameters in terms of the boost velocity  $v$ , we have

$$\mathcal{M} = \frac{M}{2} \left( \frac{2 - v^2}{1 - v^2} \right), \quad J = \frac{aM}{\sqrt{1 - v^2}}, \quad Q = \frac{Mv}{1 - v^2}.\tag{11}$$

It should be noted that the dilaton charge is not independent as it can be expressed in terms of the other parameters [20, 22]. Clearly, the ultrarelativistic limit  $v \rightarrow 1$  implies the vanishing of the “seed” (unboosted) mass  $M$  as well, thus keeping the physical mass  $\mathcal{M}$  fixed.

Another important feature of spacetime (5) arises from its dragging properties, which can easily be understood by introducing a family of locally nonrotating observers. These observers move on orbits with constant  $r$  and  $\theta$  and with a four-velocity  $u^\mu$ , obeying the condition  $u \cdot \xi_{(\phi)} = 0$ . From this condition, we find that the coordinate angular velocity of

these observers is given by

$$\Omega = -\frac{g_{03}}{g_{33}} = \frac{2aMr\sqrt{1-v^2}}{2Mr(r^2+a^2) + (1-v^2)\Delta\Sigma}. \quad (12)$$

At large distances, we have the following expansion for the angular velocity

$$\Omega = \frac{2aM}{r^3\sqrt{1-v^2}} + \mathcal{O}\left(\frac{1}{r^4}\right), \quad (13)$$

which reveals *the dragging property* of metric (5) in the  $\phi$ -direction, vanishing at spatial infinity. This expansion also confirms the physical angular momentum of the metric, given in (11). Meanwhile, as follows from equation (12), towards the event horizon the angular velocity increases, approaching its constant value at  $r = r_+$ . Thus, we have

$$\Omega_H = \frac{a}{r_+^2 + a^2} \sqrt{1-v^2}. \quad (14)$$

It is not difficult to show that the corotating Killing vector defined as  $\xi_{(t)} + \Omega_H \xi_{(\phi)}$  is tangent to the null surface of the horizon. That is, the quantity  $\Omega_H$  is nothing but the angular velocity of the horizon. We note that the angular velocity of the extreme horizon,  $a = M$ , diverges in the ultrarelativistic limit  $v \rightarrow 1$  as the horizon radius in this limit shrinks to zero, by equations (10) and (11). Therefore, in the following we will focus only on the physically acceptable values of the boost velocity, i.e. on those obeying the condition  $v < 1$ .

In summary, the spacetime metric in (5) generalizes the Kerr solution of general relativity to include the signature of the extra fifth dimension that in four dimensions shows up through the appearance of the Maxwell and dilaton fields. In other words, it describes a rotating black hole from the Kaluza-Klein point of view, whose physical properties were briefly described above.

## B. The Hamilton-Jacobi Equation

Let us now consider the geodesic motion of a massive (uncharged) particle in spacetime (5) of the Kerr-Kaluza-Klein black hole. The Hamilton-Jacobi equation governing the geodesic motion is given by

$$\frac{\partial S}{\partial \lambda} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 0, \quad (15)$$

where  $\lambda$  is an affine parameter. Since the spacetime under consideration possesses two commuting timelike and spacelike Killing vectors, one can assume that the action  $S$  admits

the following representation

$$S = \frac{1}{2} m^2 \lambda - Et + L\phi + F(r, \theta). \quad (16)$$

Here  $F(r, \theta)$  is an arbitrary function of two variables, the constants of motion correspond to the mass  $m$ , energy  $E$  and to the angular momentum  $L$  of the particle.

If we now substitute this action along with the contravariant metric components

$$\begin{aligned} g^{00} &= \frac{1}{B\Sigma} \left[ \Sigma \sinh^2 \alpha + a^2 \sin^2 \theta \cosh^2 \alpha - \frac{(r^2 + a^2)^2 \cosh^2 \alpha}{\Delta} \right], \\ g^{11} &= \frac{\Delta}{B\Sigma}, \quad g^{22} = \frac{1}{B\Sigma}, \quad g^{03} = \frac{a \cosh \alpha}{B\Sigma} \left( 1 - \frac{r^2 + a^2}{\Delta} \right), \\ g^{33} &= \frac{1}{B\Sigma} \left( \frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right), \end{aligned} \quad (17)$$

into equation (15), it is straightforward to show that the latter can be put in the form

$$\begin{aligned} &\Delta \left( \frac{\partial F}{\partial r} \right)^2 + \left( \frac{\partial F}{\partial \theta} \right)^2 + \left[ \Sigma \sinh^2 \alpha + \left( a^2 \sin^2 \theta - \frac{(r^2 + a^2)^2}{\Delta} \right) \cosh^2 \alpha \right] E^2 \\ &+ \left( \frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) L^2 - 2 a \cosh \alpha \left( 1 - \frac{r^2 + a^2}{\Delta} \right) EL = -m^2 B \Sigma. \end{aligned} \quad (18)$$

We note that separation of  $r$  and  $\theta$  variables in this equation does not occur due to the presence of the factor  $B$  on its right-hand side (the explicit form of  $B$  is given in Eq. (7)). On the other hand, such a separation occurs for the particle of zero mass ( $m = 0$ ), implying the existence of a new conserved quantity, quadratic in 4-momentum, along the null geodesics. This is due to the fact that the spacetime metric in (5) admits hidden symmetries, generated by a second rank symmetric conformal Killing tensor [2], which give rise to the new nontrivial integral of motion. Below, we explore the hidden symmetries and present the explicit form for the conformal Killing tensor.

### III. HIDDEN SYMMETRIES

We have seen that one of the salient features of the spacetime metric in (5) is that it does not allow for the complete separation of variables in the Hamilton-Jacobi equation for massive particles. Interpreting this fact in terms of pertinent hidden symmetries, one concludes that the spacetime does not admit hidden symmetries, which are generated by a



second rank symmetric Killing tensor  $K_{\mu\nu}$ , in contrast to the original Kerr spacetime. We recall that the Killing tensor obeys the equation

$$\nabla_{(\lambda} K_{\mu\nu)} = 0, \quad (19)$$

where the operator  $\nabla$  denotes the covariant differentiation and the round brackets stand for symmetrization over the indices enclosed.

Let us now assume that such a Killing tensor exists for an effective metric  $h_{\mu\nu}$ , which is conformally related to the original metric  $g_{\mu\nu}$  in (5) as follows

$$h_{\mu\nu} = e^{2\Omega} g_{\mu\nu}, \quad (20)$$

where  $\Omega$  is a smooth scalar function. It is straightforward to show that the associated Christoffel symbols  $\gamma_{\mu\nu}^\lambda$  and  $\Gamma_{\mu\nu}^\lambda$  for the metrics  $h_{\mu\nu}$  and  $g_{\mu\nu}$ , respectively, are related as

$$\gamma_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda + \left( \delta_\mu^\lambda \Omega_{,\nu} + \delta_\nu^\lambda \Omega_{,\mu} - g^{\lambda\sigma} g_{\mu\nu} \Omega_{,\sigma} \right). \quad (21)$$

Meanwhile, for the covariant derivatives of a second rank symmetric tensor  $P_{\mu\nu}$  we find the relation

$$D_{(\lambda} P_{\mu\nu)} = \nabla_{(\lambda} P_{\mu\nu)} - 4P_{(\mu\nu} \Omega_{,\lambda)} - g_{(\mu\nu} I_{\lambda)}, \quad (22)$$

where

$$I_\lambda = -2 g^{\alpha\tau} P_{\lambda\tau} \Omega_{,\alpha} \quad (23)$$

and the operator  $D$  denotes covariant differentiation with respect to the metric  $h_{\mu\nu}$ , the comma stands for the partial derivative. It is also straightforward to show that for the tensor  $P_{\mu\nu}$  defined as

$$P_{\mu\nu} = e^{-4\Omega} K_{\mu\nu}, \quad (24)$$

where  $K_{\mu\nu}$  is the Killing tensor in the metric  $h_{\mu\nu}$ , equation (22) reduces to the form

$$\nabla_{(\lambda} P_{\mu\nu)} = g_{(\mu\nu} I_{\lambda)}, \quad (25)$$

in which we recognize the defining equation for the conformal Killing tensor  $P_{\mu\nu}$  [2]. It is worth noting that for the one-form  $I = I_\mu dx^\mu$  being exact, the conformal Killing tensor goes over into the Killing tensor (see equation (19)), implying the representation

$$P_{\mu\nu} = K_{\mu\nu} + f g_{\mu\nu}, \quad (26)$$

where  $f$  is a scalar function.

With all this in mind, we turn now to the effective metric  $h_{\mu\nu}$  in (20). Choosing the function  $\Omega$  in the form

$$\Omega = -\frac{1}{2} \ln B, \quad (27)$$

we see that the Hamilton-Jacobi equation in the effective metric admits a complete separation of variables. In other words, in this case under consideration the factor  $B$  on the right-hand side of equation (18) disappears and the resulting equation allows for separation in the  $r$  and  $\theta$  variables. Thus, for the action  $S$  in the form

$$S = \frac{1}{2}m^2\lambda - Et + L\phi + S_r(r) + S_\theta(\theta), \quad (28)$$

we arrive at two independent ordinary differential equations

$$\Delta \left( \frac{dS_r}{dr} \right)^2 - \frac{1}{\Delta} \left[ (r^2 + a^2) \cosh \alpha E - aL \right]^2 + r^2 (m^2 + \sinh^2 \alpha E^2) = -K, \quad (29)$$

$$\left( \frac{dS_\theta}{d\theta} \right)^2 + \frac{1}{\sin^2 \theta} (a \cosh \alpha \sin^2 \theta E - L)^2 + a^2 \cos^2 \theta (m^2 + \sinh^2 \alpha E^2) = K, \quad (30)$$

where  $K$  is a constant of separation. It is evident that the separability occurs due to the existence of the new quadratic integral of motion  $K = K^{\mu\nu} p_\mu p_\nu$ , which is guaranteed by the existence of the irreducible Killing tensor  $K^{\mu\nu}$  in the effective metric  $h_{\mu\nu}$ , thereby confirming our assumption made above. Using this fact in equation (30) and taking into account the relation  $m^2 = -h^{\mu\nu} p_\mu p_\nu$ , we obtain the explicit form of the Killing tensor. It is given by

$$\begin{aligned} K^{\mu\nu} = & \delta_\theta^\mu \delta_\theta^\nu + a^2 \left[ \sinh^2 \alpha \cos^2 \theta + \cosh^2 \alpha \sin^2 \theta \right] \delta_t^\mu \delta_t^\nu + \frac{1}{\sin^2 \theta} \delta_\phi^\mu \delta_\phi^\nu \\ & + \left( \delta_t^\mu \delta_\phi^\nu + \delta_\phi^\mu \delta_t^\nu \right) a \cosh \alpha - a^2 \cos^2 \theta h^{\mu\nu}. \end{aligned} \quad (31)$$

For the vanishing boost parameter,  $\alpha \rightarrow 0$ , this expression agrees with that obtained in [2] for the ordinary Kerr spacetime.

Next, using equations (20) and (24) along with equation (27), we calculate the nonvanishing components of the “current-vector” in (23). As a consequence, we find that the associated current one-form is given by

$$I = \frac{Ma^2 \sinh^2 \alpha}{B\Sigma^2} \left[ (r^2 - a^2 \cos^2 \theta) \cos^2 \theta dr + r^3 \sin 2\theta d\theta \right]. \quad (32)$$

In obtaining this expression we have also used the  $K_r^r$  and  $K_\theta^\theta$  components of the Killing tensor in (31). We see that this expression vanishes for  $a = 0$  and in this case, as follows from

equation (25), the metric in (5) admits the reducible Killing tensor. However, in the general case it admits the conformal Killing tensor defined in (24). Performing some calculations, we find the explicit form for the conformal Killing tensor

$$P_{\mu\nu} = B^2 K_{\mu\nu}, \quad (33)$$

where for the Killing tensor  $K_{\mu\nu}$  is given by

$$\begin{aligned} K_{\mu\nu} dx^\mu dx^\nu = & -\frac{\Sigma}{\Delta} a^2 \cos^2 \theta dr^2 + \frac{r^2 \sin^2 \theta}{B^4 \Sigma} \left[ a \cosh \alpha dt - (r^2 + a^2) d\phi \right]^2 \\ & \frac{\Delta a^2 \sin^2 \theta}{B^4 \Sigma} \left[ \cosh \alpha dt - a \sin^2 \theta d\phi \right]^2 + r^2 \Sigma d\theta^2 + \frac{4Mr^3 \sin^2 \theta \sinh^2 \alpha}{B^4 \Sigma} \\ & \left[ (r^2 + a^2 + Mr \sinh^2 \alpha) d\phi - a \cosh \alpha dt \right] d\phi, \end{aligned} \quad (34)$$

which is obtained by lowering the contravariant indices of the tensor in (31) with respect to the metric  $h_{\mu\nu}$ . It is straightforward to verify that the conformal Killing tensor (33) satisfies equation (25) with the current-vector given in equation (32).

#### IV. GEODESIC MOTION IN THE EQUATORIAL PLANE

In this section, we restrict ourselves to the description of geodesics in the equatorial plane of the Kerr-Kaluza-Klein black hole. As we have mentioned above, such a black hole transmits the imprint of the extra fifth dimension into four-dimensional spacetime through the appearance of the electric and dilaton charges. As a consequence, its physical parameters become substantially different from those of the original Kerr black hole, as given in (11). Clearly, the effect of the extra dimension would also change the properties of observable orbits near the black hole. To get some insight into this issue, it is useful to explore the equatorial geodesic motion in metric (5). For this motion,  $\theta = \pi/2$ , from equation (18) it follows that  $\partial F / \partial \theta = 0$  and

$$\begin{aligned} F(r) = & \int \frac{dr}{\sqrt{\Delta}} \left\{ \frac{1}{\Delta} \left[ (r^2 + a^2) \cosh \alpha E - aL \right]^2 - (a \cosh \alpha E - L)^2 \right. \\ & \left. - r^2 \left( m^2 B_0 + \sinh^2 \alpha E \right) \right\}^{1/2}, \end{aligned} \quad (35)$$

where  $B_0$  denotes the value of  $B$  in (7) taken on the equatorial plane i.e.

$$B_0 = B(r, \pi/2) = \left( 1 + \frac{2M}{r} \sinh^2 \alpha \right)^{1/2}. \quad (36)$$

With this in mind, using action (16) in the equation

$$\frac{dx^\mu}{d\lambda} = g^{\mu\nu} \frac{\partial S}{\partial x^\nu} , \quad (37)$$

we obtain the following equations of motion in the equatorial plane

$$\Delta B_0 \frac{dt}{d\lambda} = \left[ \left( r^2 + a^2 + \frac{2Ma^2}{r} \right) \cosh^2 \alpha - \Delta \sinh^2 \alpha \right] E - \frac{2Ma \cosh \alpha}{r} L , \quad (38)$$

$$\Delta B_0 \frac{d\phi}{d\lambda} = \left( 1 - \frac{2M}{r} \right) L + \frac{2Ma \cosh \alpha}{r} E , \quad (39)$$

$$r^4 B_0^2 \left( \frac{dr}{d\lambda} \right)^2 = V(E, L, r, a, \alpha) , \quad (40)$$

where the effective potential in the radial equation (40) is given by

$$V = \left[ (r^2 + a^2) \cosh \alpha E - aL \right]^2 - \Delta r^2 \sinh^2 \alpha E^2 - \Delta \left[ (a \cosh \alpha E - L)^2 + B_0 m^2 r^2 \right] . \quad (41)$$

When the right-hand side of equation (40) vanishes, the geodesic motion occurs in circular orbits. The energy and the angular momentum of these orbits are given by the simultaneous solutions of the equations

$$V = 0 , \quad \frac{\partial V}{\partial r} = 0 . \quad (42)$$

Meanwhile, the region of stability of the circular orbits is governed by the inequality

$$\frac{\partial^2 V}{\partial r^2} \leq 0 , \quad (43)$$

where the case of equality refers to the innermost stable orbits.

It is worth to note that one can also provide an intriguing description of the equatorial motion in black hole spacetimes by invoking the geodesic equation

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 , \quad (44)$$

where  $\Gamma_{\alpha\beta}^\mu$  are the Christoffel symbols of the spacetime under consideration and the parameter  $s$  is supposed to be the proper time along the geodesics. Such a description possesses some simplifying advantages, in particular, when exploring the quasiequatorial motion by using the method of successive approximations. In this approach, the circular motion in the

equatorial plane is described at the zeroth-order approximation. The position four-vector of the circular orbits is given by

$$x_0^\mu(s) = \{t(s), r_0, \pi/2, \Omega_0 t(s)\}, \quad (45)$$

where  $\Omega_0$  is the orbital frequency of the motion and it is determined by the  $\mu = 1$  component of equation (44) on the equatorial plane. Meanwhile, the quasicircular or epicyclic motion occurs due to small perturbations about the circular orbits and it is the subject to the first-order approximation scheme. Substituting the associated deviation vector for small perturbations

$$\xi^\mu(s) = x^\mu(s) - x_0^\mu(s), \quad (46)$$

into the geodesic equation (44), we perform an appropriate expansion in  $\xi^\mu$ , restricting ourselves only to the first-order terms. As a consequence, we obtain the following equation

$$\frac{d^2 \xi^\mu}{dt^2} + \gamma_\alpha^\mu \frac{d\xi^\alpha}{dt} + \xi^a \partial_a U^\mu = 0, \quad a = 1, 2 \equiv r, \theta \quad (47)$$

where the quantities  $\gamma_\alpha^\mu$  and  $\partial_a U^\mu$  are taken on a circular orbit  $r = r_0$ ,  $\theta = \pi/2$  and we have passed to the coordinate time  $t$ , instead of the proper time  $s$ . After a simple algebra, we have

$$\gamma_\alpha^\mu = 2\Gamma_{\alpha\beta}^\mu u^\beta (u^0)^{-1}, \quad \partial_a U^\mu = \left( \partial_a \Gamma_{\alpha\beta}^\mu \right) u^\alpha u^\beta (u^0)^{-2}. \quad (48)$$

Next, writing down the components of equation (47), it is not difficult to show that the epicyclic motion consists of two decoupled oscillations in the radial and vertical directions, which are governed by the equations

$$\frac{d^2 \xi^r}{dt^2} + \Omega_r^2 \xi^r = 0, \quad \Omega_r = \left( \frac{\partial U^r}{\partial r} - \gamma_A^r \gamma_r^A \right)^{1/2}, \quad A = 0, 3 \equiv t, \phi, \quad (49)$$

$$\frac{d^2 \xi^\theta}{dt^2} + \Omega_\theta^2 \xi^\theta = 0, \quad \Omega_\theta = \left( \frac{\partial U^\theta}{\partial \theta} \right)^{1/2}, \quad (50)$$

respectively. It also follows that the conditions  $\Omega_r^2 \geq 0$  and  $\Omega_\theta^2 \geq 0$  determine the stability of the circular motion against small oscillations. Thus, in this framework the description of the equatorial motion in black hole spacetimes can be performed in terms of three fundamental frequencies, the orbital frequency  $\Omega_0$ , the radial  $\Omega_r$  and the vertical  $\Omega_\theta$  epicyclic frequencies.

We note that the general description of the epicyclic motion in the spacetime of stationary black holes was first given in works [36, 37]. Some details of this description can also be found

in recent works [30, 33]. Below, we calculate the physical parameters of the geodesic motion occurring in both equatorial and off-equatorial planes of spacetime (5), using the frameworks of the Hamilton-Jacobi equation as well as the geodesic equation described above.

To make the things more transparent, it is instructive to consider the cases of static and rotating black holes separately. In what follows, to figure out the effects of the extra dimension, we will express all quantities of interest only in terms of the boost velocity  $v$  and the physical mass of the black hole. This also makes the things much simpler than expressing them in terms of the electric and dilaton charges.

### A. The static case

Setting in equation (41) the rotation parameter  $a$  to zero and solving the simultaneous equations in (42), we find that the energy and the angular momentum of the circular motion around a Schwarzschild-Kaluza-Klein black hole are given by

$$\frac{E}{m} = \frac{1}{B_0^{1/2}} \frac{(r - 2M) \left(1 + \frac{3M}{2r} \sinh^2 \alpha\right)^{1/2}}{\left[r^2 - 3Mr + M(r - 4M) \sinh^2 \alpha\right]^{1/2}}, \quad (51)$$

$$\frac{L}{m} = \pm \frac{1}{B_0^{-1/2}} \frac{M^{1/2} r^{3/2} \left(B_0^2 + \cosh^2 \alpha\right)^{1/2}}{\sqrt{2} \left[r^2 - 3Mr + M(r - 4M) \sinh^2 \alpha\right]^{1/2}}. \quad (52)$$

It is not difficult to see that the region of existence of the circular motion extends from infinity up to the limiting photon orbit, whose radius is governed by the vanishing denominator of (51). Thus, in terms of the boost velocity  $v$  and the physical mass  $\mathcal{M}$ , we have the equation

$$4(r - 3\mathcal{M})r - 2(2r^2 - 11\mathcal{M}r + 8\mathcal{M}^2)v^2 + (r - 4\mathcal{M})^2 v^4 = 0 \quad (53)$$

the largest root of which is given by

$$\frac{r}{\mathcal{M}} = 4 + \frac{5}{v^2 - 2} + \frac{|v^2 - 2|}{(v^2 - 2)^2} \sqrt{9 - 8v^2}. \quad (54)$$

It follows that for  $v = 0$ , the radius of the limiting photon orbit  $r_{ph} = 3M$  as for the original Schwarzschild black hole, while it moves towards the event horizon with the growth of  $v$  and we find that for  $v \simeq 0.95$ ,  $r_{ph} \simeq 0.7\mathcal{M}$ . (We recall that for  $v = 0$ , we have  $\mathcal{M} = M$ , as follows from Eq.(11)).

It is clear that not all circular orbits in the region of existence are bound. The radius of bound circular orbits obeys the inequality  $r > r_{mb}$ , where the radius of the innermost bound orbit  $r_{mb}$  is given by the largest root of the equation

$$\left(1 - \frac{4\mathcal{M}}{r} \frac{1-v^2}{2-v^2}\right)^2 - \left(1 + \frac{4\mathcal{M}}{r} \frac{v^2}{2-v^2}\right)^{1/2} \left[1 - \frac{4\mathcal{M}}{r} \frac{1-v^2}{2-v^2} - \frac{\mathcal{M}}{r} \left(1 + \frac{4\mathcal{M}v^2}{r} \frac{1-v^2}{(2-v^2)^2}\right) \left(1 + \frac{3\mathcal{M}}{r} \frac{v^2}{2-v^2}\right)^{-1}\right] = 0. \quad (55)$$

In obtaining this equation we have used the condition  $E^2 = m^2$ , writing the result in terms of the physical mass of the black hole. We note that for  $v = 0$ ,  $r_{mb} = 4M$ , while for  $v \simeq 0.95$ , we find that  $r_{mb} \simeq 0.8\mathcal{M}$ .

As for the region of stability of the circular motion, its boundary is determined by the equation

$$1 - \frac{6\mathcal{M}}{r} \frac{2-3v^2}{2-v^2} - \frac{12\mathcal{M}^2v^2}{r^2} \frac{10v^4 - 27v^2 + 16}{(2-v^2)^3} - \frac{16\mathcal{M}^3v^4}{r^3} \frac{1-v^2}{(2-v^2)^4} - \left(31 - 22v^2 + \frac{24\mathcal{M}v^2}{r} \frac{1-v^2}{2-v^2}\right) = 0, \quad (56)$$

which is obtained by using equations (51) and (52) in (43). It follows that for  $v = 0$ , the radius of the innermost stable circular orbit  $r_{ms} = 6M$ , while for  $v \simeq 0.95$ , we have  $r_{ms} \simeq 1.3\mathcal{M}$ . We note that in the ultrarelativistic limit,  $v \rightarrow 1$ , the radius of the limiting photon orbit as well as the radii of the innermost bound and stable orbits shrink to zero, merging with the singular horizon  $r_+ = 0$ . The results of a detailed numerical analysis of equations (54)-(56) are plotted in Figure 1. We note that with increasing the boost velocity  $v$ , the regions of existence, boundedness and stability of the circular motion essentially enlarge towards the event horizon, thereby clearly showing up the physical effects of the extra fifth dimension. (In this figure and in the following ones we take  $\mathcal{M} = 1$  that makes all physical quantities of interest dimensionless).

Another quantity of physical interest is the binding energy of the innermost stable circular orbit. Using expression (51), it is not difficult to show that for  $r_{ms} \simeq 1.3\mathcal{M}$  and  $v \simeq 0.95$ , the binding energy per unit mass of a particle is

$$E_{binding} = 1 - \frac{E}{m} \simeq 0.163, \quad (57)$$

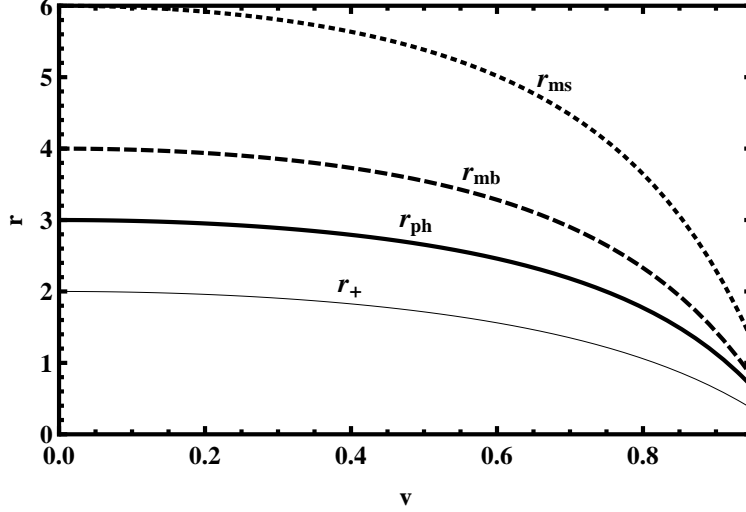


FIG. 1. Radii of circular orbits around a Schwarzschild-Kaluza-Klein black hole as functions of the boost velocity. Dotted and dashed curves indicate the positions of the innermost stable and bound orbits, whereas the solid curve refers to the limiting photon orbit. The thin curve  $r_+$  indicates the position of the event horizon.

or nearly 16.3% of the particle rest-mass energy. That is, the energy-release process in the vicinity of the Schwarzschild-Kaluza-Klein black hole is potentially much more efficient than for the original Schwarzschild black hole, for which it is about 5.72% of the rest-mass energy.

We turn now to the description of the equatorial motion in terms of the orbital and epicyclic frequencies. From the  $\mu = 1$  component of equation (44) on the equatorial plane,  $\theta = \pi/2$ , we find that the orbital frequency is given by

$$\Omega_0^2 = \Omega_s^2 \left( 1 + \frac{3\mathcal{M}}{r} \frac{v^2}{2 - v^2} \right)^{-1} \left( 1 + \frac{4\mathcal{M}}{r} \frac{v^2}{2 - v^2} \right)^{-1} f(r, \mathcal{M}, v), \quad (58)$$

where  $\Omega_s = \mathcal{M}^{1/2}/r^{3/2}$  is the Kepler frequency and

$$f(r, \mathcal{M}, v) = 1 + \frac{4\mathcal{M}v^2}{r} \frac{1 - v^2}{(2 - v^2)^2}. \quad (59)$$

It is not difficult to show that using this expression for the orbital frequency in the normalization condition for the four-velocity of the particle  $g_{\mu\nu}u^\mu u^\nu = -1$ , with equation (45) and  $E = mu_0$  in mind, we obtain the same expression for the energy of the circular motion as that given (51). Next, using equations in (49) and (50) it is straightforward to show that the vertical epicyclic frequency  $\Omega_\theta$  is precisely the same as the orbital frequency,  $\Omega_\theta^2 = \Omega_0^2$ ,



while for the radial epicyclic frequency we find that

$$\Omega_r^2 = \Omega_s^2 \left( 1 + \frac{3\mathcal{M}}{r} \frac{v^2}{2-v^2} \right)^{-1} \left( 1 + \frac{4\mathcal{M}}{r} \frac{v^2}{2-v^2} \right)^{-3} h(r, \mathcal{M}, v), \quad (60)$$

where the function  $h(r, \mathcal{M}, v)$  is the same as that given on the left-hand side of equation (56). It follows that the circular motion is always stable against small oscillations in the vertical direction ( $\Omega_\theta^2 \geq 0$ ), while the boundary of the stability region in the radial direction is given by the condition  $\Omega_r^2 = 0$ , resulting in the same equation as in (56).

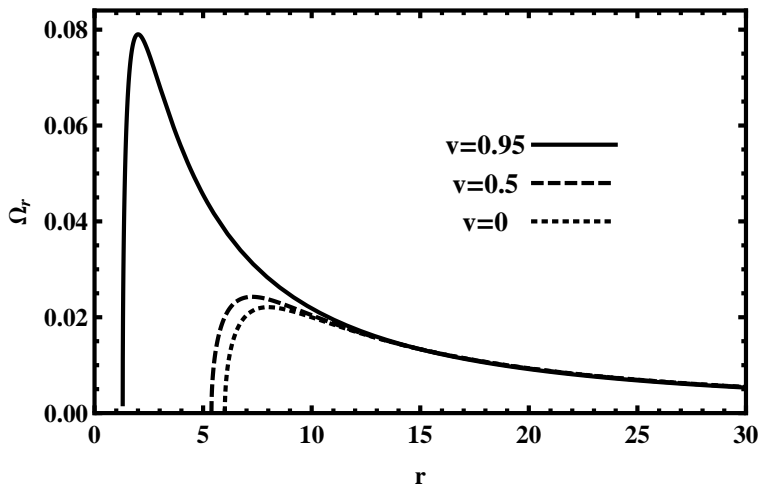


FIG. 2. The dependence of the radial epicyclic frequency on the radii of circular orbits around a Schwarzschild-Kaluza-Klein black hole for given values of the boost velocity.

In Figure 2 we plot the radial epicyclic frequency as a function of the radii of circular orbits for given values of the boost velocity. We see that with increasing the boost velocity, the location of the maximum moves towards the event horizon of the black hole. Furthermore, for the large enough value of the boost velocity, the pertaining highest value of the radial epicyclic frequency is significantly greater compared to that for the original Schwarzschild black hole,  $v = 0$ .

## B. The rotating case

In this case, the simultaneous solution of the equations in (42), determining the energy and the angular momentum of the circular motion turns out to be very formidable. Therefore, we appeal to the geodesic equation (44), in which case one gains some simplifying advantages. Substituting in this equation the Christoffel symbols for metric (5), with equation (45) in

mind, we find that its  $\mu = 0, 2, 3$  components become trivial, while the  $\mu = 1$  component yields the defining equation for the orbital frequency  $\Omega_0$  of the circular motion. Solving this equation, we obtain that

$$\Omega_0 = \frac{2\sqrt{M} \left[ -a\sqrt{M}(1+X) \pm \sqrt{XY} \right] \sqrt{1-v^2}}{r^3(1-v^2)X(1+3X) - 2a^2M(1-v^2+X)}, \quad (61)$$

where we have used the notation

$$X = 1 + \frac{2M}{r} \frac{v^2}{1-v^2}, \quad (62)$$

$$Y = 4r^3 - 2r^2v^2(r-5M) + \frac{Mv^4}{1-v^2} (3r^2 + 6Mr - a^2). \quad (63)$$

The upper sign in the numerator of (61) refers to the direct orbits (the motion of the particle is corotating with respect to the rotation of the black hole), whereas the lower sign corresponds to the retrograde, counterrotating motion of the particle. Meanwhile, from the normalization condition  $g_{\mu\nu}u^\mu u^\nu = -1$ , we find that the energy and the orbital frequency of the circular motion are related by

$$\frac{E}{m} = -\frac{g_{00} + \Omega_0 g_{03}}{\sqrt{-g_{00} - 2\Omega_0 g_{03} - \Omega_0^2 g_{33}}}, \quad (64)$$

where the components of the metric tensor are given in equation (9) with  $\theta = \pi/2$ . From this equation it follows that the radius of the limiting photon orbit is governed by the equation

$$1 - \frac{2M}{r} \left( 1 - \frac{2a\Omega_0}{\sqrt{1-v^2}} \right) - \left( r^2 + a^2 + \frac{2M}{r} \frac{r^2v^2 + a^2}{1-v^2} \right) \Omega_0^2 = 0. \quad (65)$$

Next, we substitute in this equation the expression for the orbital frequency given in (61) and express the result in terms of the physical mass of the black hole. Solving the resulting equation numerically, we find that in the limit of the extremal rotation,  $a \rightarrow M$ , and for  $v = 0$ , we have  $r_{ph} \simeq 1.23M$  ( $a = 0.98M$ ) for the direct motion, while  $r_{ph} \simeq 4M$  ( $a = M$ ) for the retrograde motion just as for an extreme Kerr black hole. On the other hand, for  $v = 0.95$  and for the the rotation parameters as given above, we find that  $r_{ph} \simeq 0.22\mathcal{M}$  and  $r_{ph} \simeq 0.92\mathcal{M}$  for direct and retrograde orbits, respectively.

It is also not difficult to show that the radius of the innermost bound orbits is given by the equation

$$\left[ 1 - \frac{2M}{r} \left( 1 - \frac{2a\Omega_0}{\sqrt{1-v^2}} \right) \right] \left[ 1 - \left( 1 - \frac{2M}{r} \right) X^{-1/2} \right] - \Omega_0^2 \left[ r^2 + a^2 + \frac{2M}{r(1-v^2)} \left( r^2v^2 + a^2 + \frac{2Ma^2}{r} X^{-1/2} \right) \right] = 0, \quad (66)$$

which is obtained from equation (64) with  $E^2 = m^2$ . Again, substituting expression (61) into this equation and performing the similar numerical analysis as in the case of (65), we find that for  $v = 0$ ,  $r_{mb} \simeq 1.50M$  (direct orbits,  $a = 0.95M$ ) and  $r_{mb} \simeq 5.83M$  (retrograde orbits,  $a = M$ ). Meanwhile, with the rotation parameters as given above and with  $v = 0.95$ , we have  $r_{mb} \simeq 0.27\mathcal{M}$  for direct orbits and  $r_{mb} \simeq 0.18\mathcal{M}$  for retrograde orbits.

As in the static case, to explore the stability of the circular motion in the radial and vertical directions we need to know the explicit expressions for the pertaining epicyclic frequencies given in equations (49) and (50). Using in equation (49) the components of the Christoffel symbols for metric (5) and performing straightforward calculations, we find that the radial epicyclic frequency is given by

$$\Omega_r^2 = \frac{1}{X^3(1-v^2)^3} \left[ \Omega_0^2 k_1 + \frac{2aM\sqrt{1-v^2}}{r^3} \Omega_0 k_2 - \frac{M}{r^3} (1-v^2) k_3 \right], \quad (67)$$

where

$$\begin{aligned} k_1 = & 3 - \frac{8M}{r} - \left( 9 - \frac{38M}{r} + \frac{39M^2}{r^2} \right) v^2 + \left( 1 - \frac{2M}{r} \right)^2 v^4 \left[ 9 - \frac{16M}{r} \right. \\ & - \left( 3 - \frac{10M}{r} + \frac{9M^2}{r^2} \right) v^2 \left. \right] - \frac{Ma^4}{r^5} \left[ 2(2-v^2)(1-v^2)^2 + \frac{3M}{r} (3-4v^2+v^4) v^2 \right. \\ & + \frac{4M^2}{r^2} v^4 \left. \right] + \frac{a^2}{r^2} \left[ \left( 1 - \frac{10M}{r} \right) (1-v^2)^3 + \frac{4M^3}{r^3} v^2 \left( 1 - 8v^2 + 5v^4 + \frac{M}{r} v^2 \right) \right. \\ & + \frac{2M^2}{r^2} (1-19v^2+31v^4-13v^6) \left. \right], \\ k_2 = & 6(1-v^2)^2 - \frac{M}{r} (2-23v^2+21v^4) - \frac{4M^2}{r^2} v^2 \left( 1 - 5v^2 + \frac{M}{r} v^2 \right) \\ & + \frac{a^2}{r^2} \left[ 4(1-v^2)^2 + \frac{9M}{r} (1-v^2) v^2 + \frac{4M^2}{r^2} v^4 \right], \\ k_3 = & 2 \left( 1 - \frac{M}{r} + \frac{2a^2}{r^2} \right) - \left[ \frac{a^2}{r^2} \left( 6 - \frac{9M}{r} \right) + \left( 3 - \frac{8M}{r} + \frac{4M^2}{r^2} \right) \right] v^2 \\ & + \left( 1 - \frac{2M}{r} + \frac{2a^2}{r^2} \right) \left( 1 - \frac{3M}{r} + \frac{2M^2}{r^2} \right) v^4. \end{aligned} \quad (68)$$

For the vanishing rotation parameter,  $a = 0$ , this expression agrees with that given in (60) for the static black hole, whereas for  $v = 0$ , it goes over into the expression for the original Kerr black hole [36, 37]. (See also works [30, 33]). The boundary of the stability region in the radial direction is determined by the equation  $\Omega_r^2 = 0$ . Writing this equation in

terms of the physical mass of the black hole, we apply a numerical analysis to explore its solutions in the extremal limit of rotation  $a \rightarrow M$ . In particular, we find that for the direct motion and for  $a = 0.95M$ , the radii of the innermost stable orbits  $r_{ms} \simeq 1.93M$  ( $v = 0$ ) and  $r_{ms} \simeq 0.4\mathcal{M}$  ( $v = 0.95$ ). Meanwhile, for the retrograde motion and for  $a = M$  we have  $r_{ms} = 9M$  ( $v = 0$ ) and  $r_{ms} \simeq 1.95\mathcal{M}$  ( $v = 0.95$ ).

For an extreme Kerr-Kaluza-Klein black hole, the results of the full numerical analysis of the boundaries of the circular motion are plotted in Figure 3. The curves clearly show that as the boost velocity increases the radius of the limiting photon orbit as well as the radii of the innermost bound and the innermost stable orbits essentially enlarge towards the event horizon, both for direct and retrograde motions.

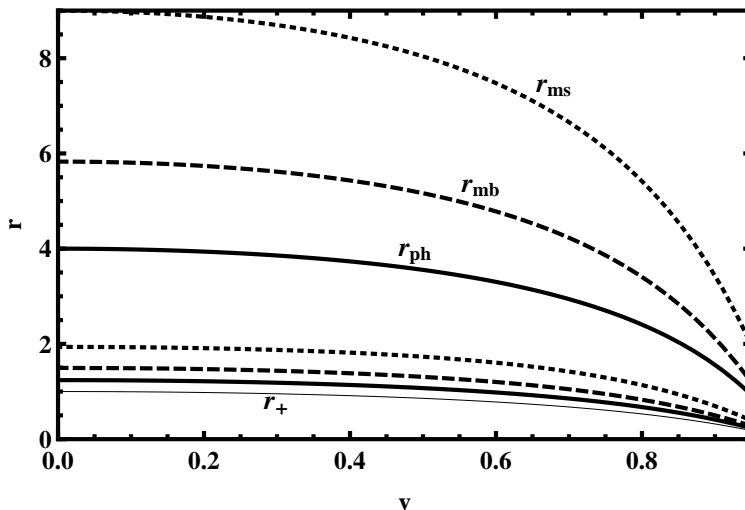


FIG. 3. Radii of circular orbits around an extreme Kerr-Kaluza-Klein black hole ( $a \rightarrow M$ ) as functions of the boost velocity. The upper set of solid, dashed and dotted curves corresponds to the limiting photon orbit, the innermost stable and the innermost bound orbits for the retrograde motion, respectively. Similarly, the lower set of solid, dashed and dotted curves refers to the limiting photon orbit, the innermost stable and the innermost bound orbits for the direct motion. The thin curve  $r_+$  indicates the position of the event horizon.

It is also of interest to explore the dependence of the radial epicyclic frequency on the radii of circular orbits. In Figure 4 we illustrate this dependence in the limit of the extremal rotation,  $a \rightarrow M$ , and for different values of the boost velocity. We see that with increasing the boost velocity, the locations of the maxima shift towards the event horizon for both direct and retrograde orbits. Accordingly, the pertaining values of the radial epicyclic frequency

become significantly higher (especially for the retrograde motion) compared to the case of the original Kerr black hole,  $v = 0$ .

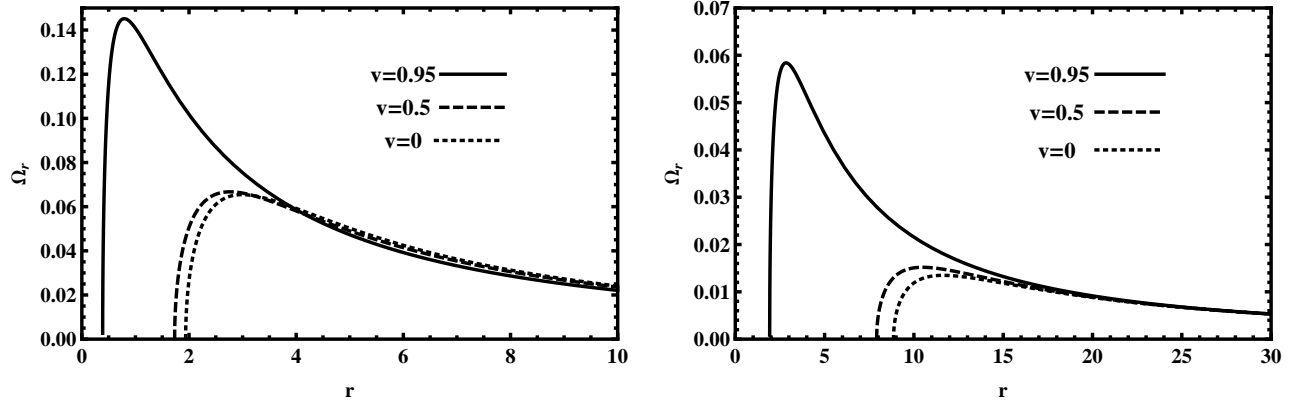


FIG. 4. The dependence of the radial epicyclic frequency on the radii of circular orbits around an extreme Kerr-Kaluza-Klein black hole for given values of the boost velocity. (*Left*: Direct orbits and  $a = 0.95M$ . *Right*: Retrograde orbits and  $a = M$ .)

Similarly, using in equation (50) the associated Christoffel symbols for metric (5) it is not difficult to show that the vertical epicyclic frequency is given by

$$\Omega_\theta^2 = \frac{1}{X^2(1-v^2)^2} \left[ \Omega_0^2 q_1 - \frac{4aM\sqrt{1-v^2}}{r^3} \Omega_0 q_2 + \frac{Ma^2}{r^5} (1-v^2) q_3 \right], \quad (69)$$

where

$$\begin{aligned} q_1 &= X(1-v^2) \left[ X(1-v^2) + \frac{a^2}{r^2} \left( 1-v^2 + \frac{M}{r} (4-v^2) \right) \right] \\ &\quad + \frac{Ma^4}{r^5} \left( 2-3v^2+v^4 + \frac{2M}{r} v^2 \right), \\ q_2 &= X(1-v^2) + \frac{a^2}{r^2} \left( 1-v^2 + \frac{M}{r} v^2 \right), \\ q_3 &= 2-v^2 + \frac{2M}{r} v^2, \end{aligned} \quad (70)$$

It is easy to show that for  $v = 0$  this expression coincides with that for the ordinary Kerr spacetime, earlier obtained in [36, 37]. A detailed numerical analysis of expression (69) shows that it is always nonnegative in the regions of existence and radial stability of the circular motion. In other words, the circular motion is stable with respect to small perturbations in the vertical direction. In Figure 5 we plot the vertical epicyclic frequency as a function of the radii of direct orbits in the field of an extreme Kerr-Kaluza-Klein black hole. We note

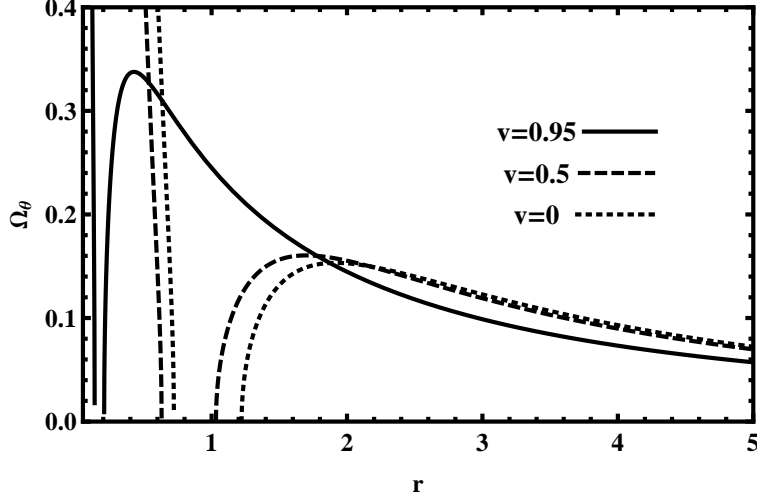


FIG. 5. The dependence of the vertical epicyclic frequency on the radii of direct circular orbits around an extreme Kerr-Kaluza-Klein black hole ( $a = 0.95M$ ) for given values of the boost velocity.

that for the direct motion, the vertical epicyclic frequency attains its highest value in the near-horizon region. Again, the growth of the boost velocity results in moving the location of the maxima towards the event horizon, thereby significantly increasing the maximum value of the vertical epicyclic frequency. It is also interesting that the location of the maxima lies in the region of the radial stability of the motion. For instance, solving the equation  $\partial\Omega_\theta/\partial r = 0$  numerically, we find that for  $a = 0.95M$  and  $v = 0.95$ , the location of the maxima is given by  $r_{max} \simeq 0.42M$ , which is greater than the radius of the pertaining innermost stable circular orbit.

To conclude this subsection, we wish to calculate the binding energies of the innermost stable circular orbits in the field of the extreme Kerr-Kaluza-Klein black hole, for both direct and retrograde motions. Using expression (64) and performing some numerical calculations, with equation (61) in mind, we find that for the direct motion

$$E_{binding} \simeq 35\%, \quad for \quad a = 0.95M, \quad v = 0.95, \quad r_{ms} \simeq 0.4M, \quad (71)$$

in contrast to the binding energy  $E_{binding} \simeq 19\%$  of a particle in the Kerr field with  $a = 0.95M$  and  $r_{ms} \simeq 1.93M$ . Similarly, for the retrograde motion we obtain that

$$E_{binding} \simeq 12\%, \quad for \quad a = M, \quad v = 0.95, \quad r_{ms} \simeq 1.95M, \quad (72)$$

while in the Kerr field with  $a = M$  and  $r_{ms} \simeq 9M$ , we have  $E_{binding} \simeq 3.7\%$ . Thus, our analysis shows that the rotating Kaluza-Klein black holes are more energetic objects,

compared to the original Kerr black holes, in the sense of the potential energy-release process in their vicinity.

## V. CONCLUSION

The remarkable property of rotating black holes in Kaluza-Klein theory is that they involve the imprint of the extra dimension through the appearance of additional charges in the spacetime metric. In the most simple setting, it is the Kerr spacetime that from the Kaluza-Klein point of view carries the signature of the extra fifth dimension by acquiring the electric and dilaton charges. In this paper, we have examined the separability structure of the Hamilton-Jacobi equation for geodesics and the pertaining hidden symmetries in the spacetime of the Kerr-Kaluza-Klein black hole. We have shown that in the general case of massive geodesics, the Hamilton-Jacobi equation does not admit the complete separation of variables, whereas such a separability occurs for massless geodesics. This fact implies the existence of hidden symmetries in the spacetime, which are generated by a second rank conformal Killing tensor. Next, we have employed a simple framework based on the effective metric which has the following properties: (i) it is conformally related to the original spacetime metric under consideration, (ii) it admits the Killing tensor, rendering the associated Hamilton-Jacobi equation for massive geodesics completely separable. With this framework, we have constructed the explicit expression for the conformal Killing tensor.

We have also examined the properties of the geodesic motion in the equatorial plane of the Kerr-Kaluza-Klein black holes, using the frameworks of both the Hamilton-Jacobi and geodesic equations. In order to make the description more transparent, we have considered the cases of static and rotating black holes separately. For both cases, we have obtained the analytical expressions for the energy and angular momentum/orbital frequency of the circular motion as well as we have derived the defining equations for the boundaries of the regions of existence, boundedness and stability of the motion. In order to gain some simplifying advantages, we have also invoked the description of the geodesic motion in terms of three fundamental frequencies: The orbital frequency, the radial and vertical epicyclic frequencies and we have obtained the associated analytical expressions for these frequencies. Next, applying a numerical analysis, we have found that the greatest effect of the extra fifth dimension amounts to the significant enlarging of the regions of existence, boundedness and

stability towards the event horizon, regardless of the classes of orbits. Furthermore, it turns out that for the large enough values of the boost velocity, the locations of the maxima of the epicyclic frequencies essentially shift towards the event horizon, thereby resulting in much greater values of these frequencies, compared to those for the original Schwarzschild/Kerr black holes, respectively.

Finally, we have explored the binding energy of the innermost stable circular orbits for both the static and rotating Kaluza-Klein black holes. It is interesting that for these black holes the energy-release process in their vicinity turns out to be potentially much more efficient than for the ordinary Schwarzschild and Kerr black holes of general relativity. It should be emphasized that throughout the paper we have focused on the physical aspects of our description. Of course, it would also be of interest to explore possible astrophysical implications of our results, especially in the context of high frequency quasiperiodic oscillations observed in some black hole binaries. This is an intriguing task for future work.

## VI. ACKNOWLEDGMENTS

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